Affine Grassmanian 1

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Def Let $R \in \mathrm{Alg}_{\mathbb{C}}$.

$\mathbb{D}_R := \operatorname{Spec} R[[t]]$ "formal disc"

$\mathbb{D}_R^\times := \operatorname{Spec} R((t))$ "formal punctured disc"

Def Let G be an algebraic group/$\mathbb{C}$.

Define functors $G[[t]], G((t)) : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Set}$

$G[[t]](R) := G(\mathbb{D}_R) = \mathrm{Hom}_{Sch}(\mathbb{D}_R, G)$

$G((t))(R) := G(\mathbb{D}_R^\times) = \mathrm{Hom}_{Sch}(\mathbb{D}_R^\times, G)$

Def The Affine Grassmanian is the functor $G((t))/G[[t]] : \mathrm{Alg}_{\mathbb{C}} \longrightarrow \mathrm{Set}$

Def A principal G-bundle on X is a fiber bundle $\pi : P \to X$ w/ a right action of G on P s.t.

(1) G acts freely transitively on fibers

(2) G preserves fibers

Ex (trivial G-bundle): $P = X \times G \to X$

$(x,g)g' := (x, gg')$

Remark: For $G = GL_n$

$\{\text{principal } GL_n\text{-bundles}\} \simeq \{\text{vector bundles}\}$

Def Gr_G is the functor: $\mathrm{Alg}_{\mathbb{C}} \to \mathrm{Set}$

$$Gr_G(R) = \left\{ (P, \sigma) \;\middle|\; \begin{array}{l} \bullet\ P \to \mathbb{D}_R \text{ is a principal } G\text{-bundle} \\ \bullet\ \sigma : P|_{\mathbb{D}_R^\times} \xrightarrow{\sim} \mathbb{D}_R^\times \times G \end{array} \right\} \Big/ \simeq$$

Affine Grassmanian 2

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Prop: There is a natural isomorphism

$$G((t))/_{G[[t]]}(R) \simeq Gr_G(R)$$

Pf Sketch: Restrict to $G = GL_n$, $R = \mathbb{C}$. Then $\mathbb{C}[[t]]$ is a PID $\Rightarrow$ all projective (aka locally free) modules are free (trivial)

$$\Rightarrow \text{RHS} = \left\{ \begin{array}{c} \mathbb{D}_{\mathbb{C}} \times G \\ \downarrow \\ \mathbb{D}_{\mathbb{C}} \end{array}, \ \theta : \mathbb{D}_{\mathbb{C}} \times G|_{\mathbb{D}^\times_{\mathbb{C}}} \simeq \mathbb{D}^\times_{\mathbb{C}} \times G \right\}$$

Lem: $\mathrm{Aut}_X(X \times G) = \mathrm{Hom}_{Sch}(X, G)$

$$\Rightarrow \text{LHS} = \left\{ \gamma : \mathbb{D}^\times_{\mathbb{C}} \times G \to \mathbb{D}^\times_{\mathbb{C}} \times G \right\} / \mathrm{Aut}_{\mathbb{D}_{\mathbb{C}}}(\mathbb{D}_{\mathbb{C}} \times G)$$

$\simeq$ ◻

Thrm: Gr_G is represented by an ind-scheme (filtered colimit of schemes where transition maps are closed immersions)

Ex ($G = G_a = \mathrm{Spec}\, \mathbb{C}[x]$)

$$Gr_{G_a}(R) = \frac{\mathrm{Hom}_{alg}(\mathbb{C}[x], R((t)))}{\mathrm{Hom}_{alg}(\mathbb{C}[x], R[[t]])} = \frac{R((t))}{R[[t]]}$$

$$= \frac{\left\{ \sum_{i \geq -n}^{\infty} r_i t^i \mid r_i \in R \right\}}{\left\{ \sum_{i=0}^{\infty} r_i t^i \mid r_i \in R \right\}} = \varinjlim \mathbb{A}^n_{\mathbb{C}}$$

Remark: Don't lose too much if just work with $Gr_G(\mathbb{C})$. Let $K = \mathbb{C}((t))$, $O = \mathbb{C}[[t]]$

$$Gr_G(\mathbb{C}) = \frac{G(K)}{G(O)}$$ aka matrices w/ coefficients in K/O

Raviolo and Loop Rotation

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Q: What is moduli interpretation of

$$G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})$$

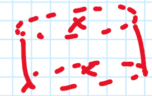
A: Recall $-G(\mathcal{K}) = \mathrm{Aut}(\mathbb{D}_{\mathbb{C}}^{\times} \times G)$

- all principal G bundles on $\mathbb{D}_{\mathbb{C}}$ are trivial

$\Rightarrow$ the datum of a principal G-bundle on $\mathrm{Rav} = \mathbb{D}_{\mathbb{C}} \cup_{\mathbb{D}_{\mathbb{C}}^{\times}} \mathbb{D}_{\mathbb{C}}$ = transition function on $\mathbb{D}_{\mathbb{C}}^{\times}$ = element of $G(\mathcal{K})$

$\Rightarrow$ principal G-bundles$/$iso $\Longleftrightarrow$ changing the trivial bundle on each $\mathbb{D}_{\mathbb{C}}$ by auto

$$\Rightarrow G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}) = \{\text{principal } G \text{ bundles on } \mathrm{Rav}\}/\simeq$$

Rem: Rav "=" alg-geo replacement for S^2 

Def (loop rotation) $\varphi_c : R[[t]] \longrightarrow R[[t]]$ where $\varphi_c(t) = ct$, $c \in R^{\times}$, is a ring automorphism

$\Rightarrow \exists$ action of $\mathbb{G}_m \curvearrowright Gr_G$

$$\begin{tikzcd}
(\varphi_c^* P, \varphi_c^* \sigma) \arrow[r] \arrow[d] & (P, \sigma) \arrow[d] \\
\mathbb{D}_R \arrow[r, "\varphi_c"] & \mathbb{D}_R
\end{tikzcd}$$

$$c \cdot (P, \sigma) = (\varphi_c^* P, \varphi_c^* \sigma)$$

Ex: For $Gr_G(\mathbb{C})$, loop rotation will be map, $c \in \mathbb{C}^{\times}$

$$c \cdot g(t) = g(ct) \text{ "=" } \begin{pmatrix} g_{11}(ct) & \cdots & \\ \vdots & & \\ & & g_{nn}(ct) \end{pmatrix}$$

Affine Flag Variety

Now $G = GL_n$

<u>Def</u> A lattice Λ in K^n is a free $\mathcal{O}$ submodule of K^n s.t. $\Lambda \otimes_{\mathcal{O}} K \cong K^n$

<u>Ex (standard lattice)</u>: $\Lambda^0 = \mathcal{O}^n$

<u>Prop:</u> $Gr_{GL_n}(\mathbb{C}) \simeq \{ \Lambda \mid \Lambda \text{ is a lattice in } K^n \}$!!

<u>Pf:</u> Consider $GL_n(K) \circlearrowright Latt_n$

- is transitive since Λ is free $\mathcal{O}$ submodule
- $Stab_{GL_n(K)}(\Lambda^0) = GL_n(\mathcal{O})$

<u>Ex:</u> For lattices, loop rotation will be

$$c \cdot \Lambda = \Lambda \otimes_{\mathbb{C}[[t]], c} \mathbb{C}[[t]]$$

Affine Flag Variety

<u>Def</u> $\pi : \mathbb{C}[[t]] \to \mathbb{C}$, $\pi(t) = 0$ induces map $G(\mathcal{O}) \xrightarrow{G(\pi)} G(\mathbb{C})$. Let $B = B(\mathbb{C})$, then the Iwahori subgroup of $G(\mathcal{O})$ is

$$I = G(\pi)^{-1}(B)$$

<u>Ex ($G = GL_2$)</u> $I = \left\{ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix} \middle| \text{ invertible} \right\}$

<u>Def</u> $$AFl_G := G(K)/I$$

<u>Rem:</u> By construction
$$\begin{array}{ccc} G/B & \longrightarrow & AFl_G \\ & & \downarrow \\ & & Gr_G \end{array}$$

<u>Def:</u> A full periodic chain in K^n is

$$\Lambda^\bullet = (\Lambda_0 \supset \Lambda_1 \supset \dots \supset \Lambda_{n-1} \supset \underset{=\Lambda_n}{t\Lambda_0})$$

s.t. $\Lambda_i \in Latt_n$, $\dim_{\mathbb{C}} \Lambda_i / \Lambda_{i+1} = 1$

Affine Flag Variety

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Ex (standard periodic chain):

$$(\Lambda^0)^\bullet = \Big(\Lambda_0 = \bigoplus_{j=0}^{n-1} \mathcal{O} e_{j+1} \supset$$

$$\Lambda_1 = \bigoplus_{j=0}^{n-2} \mathcal{O} e_{j+1} \oplus t\mathcal{O} e_n \supset$$

$$\vdots$$

$$\Lambda_n = \bigoplus_{j=0}^{n-1} t\mathcal{O} e_{j+1} \Big)$$

Prop: $G(k)/I = \{ \Lambda^\bullet \mid \Lambda^\bullet$ is a full periodic chain in $k^n \}$

Pf: $G(k) \curvearrowright$ PCLattn $\cdot$ transitively why?

Lem: All elements of PCLattn look like $(\Lambda^0)^\bullet$ where $\{e_i\}_{i=1}^n$ is any (ordered) basis of k^n

pf sketch: (1) $0 \to \frac{\Lambda_1}{t\Lambda_0} \to \frac{\Lambda_0}{t\Lambda_0} \to \frac{\Lambda_0}{\Lambda_1} \to 0$

(2) $0 \to \frac{t\Lambda_0}{t\Lambda_1} \to \frac{\Lambda_1}{t\Lambda_1} \to \frac{\Lambda_1}{t\Lambda_0} \to 0$

(3) Nakayama's lemma

$\Rightarrow$ PCLattn $\simeq \{$ ordered basis of $k^n\}$ $\curvearrowleft GL_n(k)$ transitive

- $\mathrm{Stab}_{G(k)} (\Lambda^0)^\bullet = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \\ t\mathcal{O} & \mathcal{O} & \cdots \\ \vdots & t\mathcal{O} & \end{pmatrix} = I$

(look at $(\Lambda^0)^\bullet$ mod t)

Def The (extended) affine Weyl group is

$$\widetilde{W}^{ext} := N_{G(k)}(T(k)) / T(\mathcal{O})$$

Lem: $\widetilde{W}^{ext} \simeq X_*(T) \rtimes W = \mathbb{Z}^n \rtimes S_n$

Pf: same proof as finite shows

$$N_{G(k)} T(k) / T(k) = \text{permutation matrices} = S_n$$

Affine Weyl Group

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$a_i \in \mathbb{C}^\times$

- when switching $T(k)$ w/ $T(\mathcal{O})$, $\mathcal{O}^\times = a_i + t\mathcal{O}$

$\det\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} = t^n \notin \mathcal{O}^\times$ for $n \neq 0 \in \mathbb{Z}$

$\Rightarrow \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \notin T(\mathcal{O})$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$

$\Rightarrow t^\lambda = \begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix}$ are now all distinct rep in $\widehat{W}^{ext}$

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & t^n \end{pmatrix} = s_{(12)}\begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \widehat{W}^{ext} = \mathbb{Z}^n \rtimes S_n$

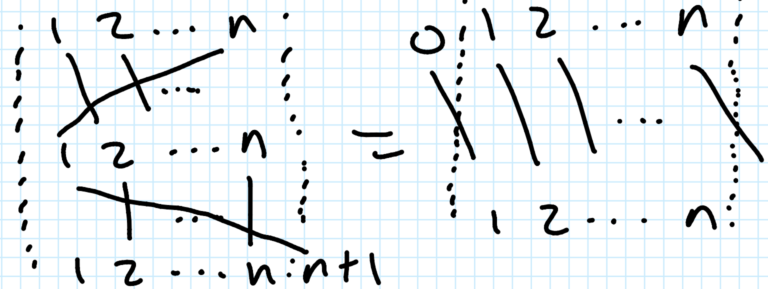
Prop $\mathbb{Z}^n \rtimes S_n = \left\{ \theta : \mathbb{Z} \to \mathbb{Z} \;\middle|\; \begin{array}{l} \cdot\ \theta \text{ is a bijection} \\ \cdot\ \theta(k+n) = \theta(k) \end{array} \right\}$

Pf: θ det by $\{1, \dots, n\}$. S_n is usual

$t^\lambda(i) = i + \lambda_i n$, $i \in \{1, \dots, n\}$

$\Rightarrow$ can draw $\widehat{W}^{ext}$ as strand diagrams on a cylinder

Ex: $\pi = ((1, 0, \dots, 0), [2\,3 \dots n\,1])$



Lem: For $w \in S_n$, $\ell(w) :=$ length of a reduced exp for w

$\ell(w) = \#$ of crossings in a strand diagram for w where each strand can cross each other at most once

Def For $(t^\lambda, w) \in \widehat{W}^{ext}$

$\ell(t^\lambda, w) =$

Remark: $\exists\, w$ s.t. $\ell(w) = 0$, but $w \neq id$

$\ell(\pi) = \ell(\pi^n) = 0 \quad n \in \mathbb{Z}$

$\{ w \mid \ell(w) = 0 \} \cong \pi_1(G(\mathbb{C}))$

Affine Convolution Algebra

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Prop (Iwahori-Bruhat decomp)

$$G(K)/I = \coprod_{w \in \widehat{W}^{ext}} IwI/I$$

$$I_w = IwI/I \cong \mathbb{A}_{\mathbb{C}}^{\ell(w)}, \quad \overline{I_w} = \coprod_{y \leq w} I_y$$

- Recall $\mathbb{C}^\times \curvearrowright G(K)/I$ by loop rotation
- $G(K) \curvearrowright G(K)/I$ by left multiplication

$\Rightarrow G(K) \rtimes \mathbb{G}_m \curvearrowright G(K)/I$ via

$$(g(t), c) \cdot h(t) = g(t)h(ct), \quad g(t), h(t) \in G(K)/I, \ c \in \mathbb{C}^\times$$

Q: what is

$$R = H_*^{BM, G(K) \rtimes \mathbb{C}^\times}(AFl_G \times AFl_G) \ ?$$

- For an ind-scheme $X = \varinjlim_i X_i$ proper

$$H_*^{BM}(X) := \varinjlim_i H_*^{BM}(X_i)$$

- Recall

$H_*^{BM,G}(G/B \times G/B) = NH_G$ is gen by Demazure op $[\overline{O_w}]_{w \in W}$ + action of $H_*^{BM,G}(G/B) = H_T^*(pt)$

- Sketchy alg:

$$G(K) \backslash G(K)/I \times G(K)/I = I \backslash G(K)/I$$

$$\Rightarrow \Phi : H_*^{BM, G(K) \rtimes \mathbb{C}^\times}(AFl_G \times AFl_G) \cong H_*^{BM, I \rtimes \mathbb{C}^\times}(AFl_G)$$

$\Rightarrow$ For $w \in \widehat{W}^{ext}$, let

$$\delta_w = \Phi^{-1}([\overline{I_w}]) \in R$$

Relations in the Convolution Algebra

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Thrm R is gen by $\{\delta_w\}_{w\in\widetilde{W}^{ext}}$ and action of $H_*^{BM,\, G(k)\rtimes\mathbb{C}^*}(G(k)/I) = H^*_{I\rtimes\mathbb{C}^*}(pt)$

$$\underset{uni}{=} H^*_{B\rtimes\mathbb{C}^*}(pt) \underset{uni}{=} H^*_{T\rtimes\mathbb{C}^*}(pt) = \mathbb{C}[y_1,\dots,y_n,\hbar]$$

Relations: (1) NilHecke Relations

$$\delta_w\delta_{w'} = \begin{cases}\delta_{ww'} & \ell(w)+\ell(w')=\ell(ww') \\ 0 & \text{otherwise}\end{cases}$$

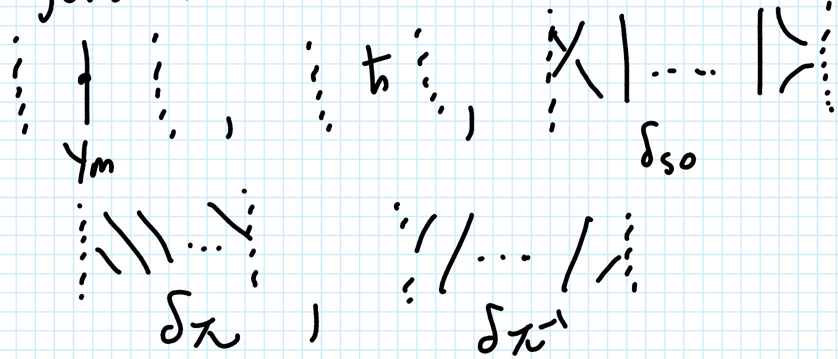
(2) Polynomial sliding

$$\delta_w\circ f = w(f)\circ\delta_w \quad \text{if } \ell(w)=0$$

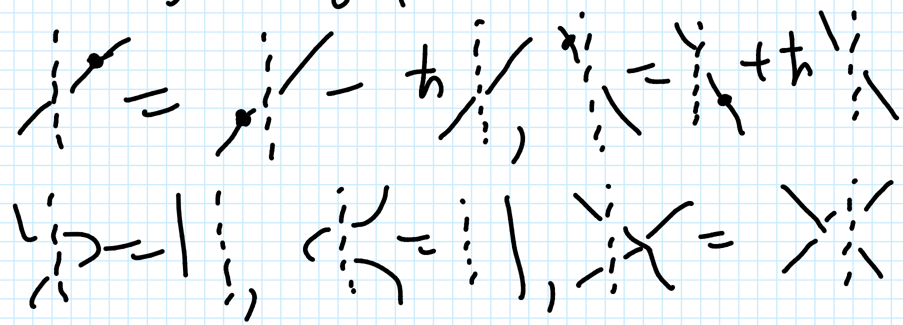
$$\delta_{s_i}\circ f = s_i(f)\circ\delta_{s_i} + \frac{s_i f - f}{\alpha_i} \quad \alpha_i\text{: simple root}$$

Diagramatically,

- generators



- Relations: NH_G +



Coulomb Branch

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Def $A = H_*^{BM, G(\mathcal{K}) \rtimes \mathbb{C}^\times}(Gr_G \times Gr_G)$

is the quantum coulomb branch

for $(G, \{0\})$

Problem: We only understand

$R = H_*^{BM, G(\mathcal{K}) \rtimes \mathbb{C}^\times}(AFl_G \times AFl_G)$

Solution: Decomp theorem

$\Rightarrow R = A \oplus \cdots$ A is subalg of R

$\Rightarrow A = (R, e)$ e idempotent in R

$\Rightarrow$ Can compute relations if one identifies e explicitly.

– Ben does this in the notes